Fall 09

Solution final fall 09

— Steve Khoury !

1) $y' = \dfrac{-(y^2+2xy)}{x^2+2xy-1}$

$dy\underbrace{(x^2+2xy-1)}_{N} + dx\underbrace{(y^2+2xy)}_{M} = 0$

Exact !

$M_y = 2y+2x \quad N_x = 2x+2y$

so $\int N\,dy = x^2y + xy^2 - y + g(x) = U$

we want $g(x)$!

so $\dfrac{dU}{dx} = M = 2xy + y^2 + g'(x) = 2xy + y^2$

so $g'(x) = 0$

$g(x) = C$

so

$$\boxed{x^2y + xy^2 - y + g(x) = C}$$

2) $y'' + 2y' - 3y = 0 \rightarrow$ Easy High order ode

$\begin{cases} y(0) = 4 \\ y'(0) = 0 \end{cases}$

3) $x^2y'' - 2xy' + 2y = x^2$ using VOP

$t^2 - t - 2t + 2 = 0$

$t^2 - 3t + 2 = 0$ so $y(1) = c_1 x + c_2 x^2$

$(t-1)(t-2) = 0$

so $y_1 = x \quad y_2 = x^2$

so $u_1 = \int \frac{-y_2 \cdot F(x)}{w(x, x^2)}$ $u_2 = \int \frac{y_1 \cdot F(x)}{w(x, x^2)}$

$F(x) = 1 = \frac{x^2}{x^2}$ $w(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$

$u_1 = \int -\frac{\cancel{x^2} \cdot 1}{\cancel{x^2}} = -x$ $u_2 = \int \frac{x}{x^2} = \int \frac{1}{x} = \ln(x)$

so $y = u_1 \cdot y_1 + u_2 \cdot y_2 + C_1 y_1 + C_2 y_2$

$$\boxed{y = -x^2 + \ln(x) \cdot x^2 + C_1 x + C_2 x^2}$$

4) $t^2 y'' + 2t y' - 2y = 0$

a) $y_1(t) = \frac{1}{t^2}$ solution ?

$y_1' = -2t^{-3}$ $y_1'' = 6t^{-4}$ $\Big\}$ replace $6t^{-2} + 2t(-2t^{-3}) - 2t^{-2} = 0$ ✓

b) Reduction of order so

Let's $\boxed{y_2 = u \cdot y_1}$ so resolve

here $p_1(x) = \frac{2}{x}$!

$u'' \cdot y_1 + u'(2y_1' + p_1(x) \cdot y_1) = 0$

$u'' \cdot \frac{1}{t^2} + u'\left(-\frac{4}{t^3} + \frac{2}{t} \cdot \frac{1}{t^2}\right) = 0$

$u'' + u'\left(-\frac{4}{t} + \frac{2}{t}\right) = 0$

$u'' + u'\left(-\frac{2}{t}\right) = 0$

Integrating factor

(2)

$$e^{\int -\frac{2}{t}} = t^{-2}$$

so $u'' \cdot t^{-2} = C$

$u'' = C.t^2$ but $u y_1 = y_2$

so $u = \frac{C}{3}.t^3 + C_1$

so $y_2 = \frac{C}{3} t^3 \left(\frac{1}{t^2}\right) + C_1 \left(\frac{1}{t^2}\right)$

$$\boxed{y_2 = \frac{C}{3} t + \frac{C_1}{t^2}}$$

$$\boxed{y_1 = \frac{1}{t^2}}$$

5) $y''' - y'' = e^t$ $\begin{cases} y(0) = y'(0) = 0 \\ y''(0) = 1 \end{cases}$

$L[y'''] = s^3 L[s] - s^2 \cancel{y(0)} - s \cancel{y'(0)} - y''(0)$

$= s^3 L[s] - 1$

$L[y''] = s^2 L[y] - s \cancel{y(0)} - \cancel{y'(0)}$

so $L[e^t] = \frac{1}{s-1}$

so $L[s](s^3 - s^2) = \frac{1}{s-1} + 1$

So

$$L[s] = \frac{1}{(s-1)(s^3-s^2)} + \frac{1}{s^3-s^2}$$

$$L[s] = \underbrace{\frac{1}{(s-1)s^2(s-1)}}_{1} + \frac{1}{s^2(s-1)} \quad (2)$$

(2) So $\frac{1}{s^2(s-1)} = \frac{As+B}{s^2} + \frac{C}{s-1} = \frac{-s+1}{s^2} + \frac{1}{s-1}$

so $(A+C)s^2 - B + As + Bs = 1$

$B = 1 \quad A = -1 \quad C = 1$

So $= -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s-1}$

$$\boxed{F_1(t) = -1 + t + e^t}$$

(1) $\frac{1}{(s-1)(s-1)s^2} = \frac{As+B}{s^2+1-2s} + \frac{Cs+D}{s^2}$

So $As^3 + Bs^2 + Cs^3 + Cs - 2Cs^2 + Ds^2 + D - 2Ds = 1$

$A + C = 0$
$B - 2C + D = 0$
$C - 2D = 0$
$D = 1$

So $\begin{cases} D = 1 \\ C = 2 \\ B = 3 \\ A = -2 \end{cases}$

So $\dfrac{-2s+3}{(s-1)^2} + \dfrac{2s+1}{s^2}$

$$= \frac{-2\left(s+\frac{3}{2}\right)}{(s-1)^2} + \frac{2}{s} + \frac{1}{s^2}$$

$$= -2\left[\frac{s-1+\frac{5}{2}}{(s-1)^2}\right] + \frac{2}{s} + \frac{1}{s^2}$$

$$= -2\left[\frac{1}{(s-1)} + \underbrace{\frac{5}{2}\cdot\frac{1}{(s-1)^2}}_{(11)}\right] + \frac{2}{s} + \frac{1}{s^2}$$

so $f_2(t) = -2\left[e^{-t} + \frac{5}{2}t\cdot e^{t}\right] + 2 + t$

so $\boxed{y = f_1(t) + f_2(t)}$

6) $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda-2 & 1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda-3 \end{bmatrix} = (\lambda-3)(\lambda-1)^2$

$\lambda = 3$ multiplicity 1 $\longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 1$ multiplicity 2 $\longrightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ & ?

Defective Matrix !

Rank = 3 $\quad A \to$ Row Reduce $\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7)

$$x' = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} x$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow$$ 2 lines that are related to $\lambda = 1$!

$$P = (v_1, v_2, v_3)$$

we want $P^{-1} A P = D$

so $AP = PD$

so $A(v_1, v_2, v_3) = (v_1, v_2, v_3) \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{cases} A v_1 = 3 v_1 \\ A v_2 = v_2 \\ A v_3 = v_2 + v_3 \end{cases}$$

so $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\leftarrow$ no choice

$v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

so $(A - I \cdot 1) v_3 = v_2$

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

!

so $x - y = 1$, $z = 0$ so $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = v_3$

or $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$?

so $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$x' = A x$

so $x = Pz$

so $z' = P^{-1} \cdot A \cdot P \cdot z$

$$z' = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

so $z_1' = 3z_1$ so $z_1 = e^{3x} \cdot c_1$

$z_2' = z_2 + z_3$ $z_3 = c_3 e^{x}$

$z_3' = z_3$ $z_2' - z_2 = c_3 e^{x}$

so $z_2 \cdot e^{-x} = \int c_3 \, dx$

$z_2 = e^{x} (c_3 x + c_2)$

so
$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} c_1 e^{x} \\ e^{x}(c_3 x + c_2) \\ c_3 e^{x} \end{pmatrix}$$

so
$$x = Pz = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 e^{x} \\ e^{x}(c_3 x + c_2) \\ c_3 e^{x} \end{pmatrix}$$

So
$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} c_1 e^{x} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{x} (c_3 x + c_2) + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} c_3 e^{x}$$

if you verify it's working ✓

8)

function of t = function of x $\quad x = t$

$x' = Ax + B$ & $b = \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix}$ & $x(0) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad x = Pz \;!$$

So

$$z' = \left(P^{-1} \cdot A \cdot P\right) z + P^{-1} \cdot B$$

Here $\quad P^{-1} b = \underset{(3\times3)}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}} \underset{(3\times1)}{\begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix}}$

$$z' = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} z + \begin{pmatrix} 1 \\ t^2 \\ t^2 - t \end{pmatrix}$$

Develop

So $\begin{cases} z_1' = 3z_1 + 1 \\ z_2' = z_2 + z_3 + t^2 \\ z_3' = z_3 + t^2 - t \end{cases}$

So $\cdot\, z_1' - 3z_1 = 1 \qquad z_1 \cdot e^{-3t} = \int e^{-3t}\, dx$

$$\boxed{z_1 = -\frac{1}{3} + C_1 \cdot e^{3t}}$$

$\cdot\, z_3' - z_3 = t^2 - t$

$z_3 \cdot e^{-t} = \int t^2 e^{-t} - t e^{-t}\, dt \Big\}$ integration by parts

So $\boxed{z_3 = C_3\, e^{t} - t^2 - t - 1}$

$$z_2' - z_2 = \left(c_3 e^t - \cancel{t^2} - t - 1\right) + \cancel{t^2}$$

$$z_2' - z_2 = c_3 e^t - t - 1$$

so $$z_2 \cdot e^{-t} = \int \left(c_3 - t e^{-t} - e^{-t}\right) dt$$

$$z_2 \cdot e^{-t} = c_3 \cdot t - e^{-t}(-t-2)$$

$$\boxed{z_2 = c_3 \cdot t \cdot e^t + t + 2 + c_2 e^t}$$

so
$$x = Pz = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1/3 + c_1 e^{3t} \\ c_3 t \cdot e^t + t + 2 + c_2 e^t \\ c_3 e^t - t^2 - t - 1 \end{pmatrix}$$

so $$x(0) = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1/3 + c_1 \\ c_2 + 2 \\ c_3 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 - 1/3 \\ c_2 + 2 \\ c_3 - 1 \end{pmatrix}$$

so $$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_2 + 2 \\ c_2 - c_3 + 1 + 2 \\ c_1 - \frac{1}{3} \end{pmatrix}$$

$1 - c_2 - 3 = -c_3$

$-2 + 4 =$

so $$\begin{cases} c_1 = 4/3 \\ c_2 = -4 \\ c_3 = -2 \end{cases}$$

So

$$x(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1/3 + 4/3 \cdot e^{3t} \\ -2te^{t} + t + 2 - 4e^{t} \\ -2e^{t} - t^2 - t - 1 \end{pmatrix}$$

verify your result!!! if it is right you have the points (for sure) for 7/8!!

but start & verify with 7!

So ~~we~~ get

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2te^t + t + 2 - 4e^t \\ -2te^t + t + 2 - 4e^t + 2e^t + t^2 + t + 1 \\ -\frac{1}{3} + \frac{4}{3} \cdot e^{3t} \end{pmatrix}}$$

So
$$\begin{cases} x_1' = -2te^t - 2e^t + 1 - 4e^t \\ x_2' = -2te^t - 2e^t + 1 - 4e^t + 2e^t + 2t + 1 \\ x_3' = 4e^{3t} \end{cases}$$

So we should replace then in

$$x' = Ax + b = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix}$$

So x_1' should be equal to $t^2 + 2x_1 - x_2$!

x_2' to $x_1 + t$

x_3' to $3x_3 + 1$ ✓ ok

most ⚠ if working evrt should be good !

$$t^2 + 2x_1 - x_2 = -4te^t + 2t + \cancel{4} - 8e^t + 2te^t - t - \cancel{2} + 4e^t$$
$$- 2e^t - t^2 - t - 1 - \cancel{t^2}$$
$$= -2te^t + 1 - 6e^t \to x_1'$$

So Good !